IMPRESSION OF A PUNCH ON A LINEARLY-DEFORMING FOUNDATION TAKING INTO ACCOUNT FRICTION FORCES

(VDAVLIVANIE SHTAMPA V LINEINO-DEFORMIRUEMOE OSNOVANIE S UCHETOM SIL TRENIIA)

PMM Vol. 31, No. 2, 1967, pp. 337-343

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(Received June 9, 1966)

An approximate method is proposed for solving (Sections 1, 4) the plane problem concerned with the impression of a punch (taking into account Coulomb friction forces) on a linearly deforming foundation of a more general type than the one investigated in [1 and 2]. The method is essentially based on a new integral relationship which is derived herein (Section 2) for Jacobi polynomials.

In the process, we also obtain the solution (Section 3) to an integral equation which is more general than the one arising from the plane contact problem which takes into account friction forces and for which the modulus of elasticity of the half-space is a power function: it is also more general than the integral equation arising from the contact problem investigated by Arutiunian and Manukian [3]. The solution is obtained by a more direct and elementary method than the one used by the above mentioned authors.

1. Consider a linearly deforming foundation subjected to a compressive force P applied through a rigid punch of width l (we are considering a plane problem) and surface shape given by $y = g(x)$. In addition, the punch is subjected to a sliding force $T = kP$, where k is the coefficient of friction between the punch and the foundation. The problem is to find $p(x)$ and $q(x)$, the normal and shear contact stresses, respectively, assuming that the contact surface is equal to the punch width and that $q(x) = kp(x)$. For the mathematical formulation of this problem, it is necessary that we know the vertical displacements of the surface points of the foundation

$$v_0^*(x) = \theta_1 v_0(x), \qquad v_1^*(x) = \theta_2 v_1(x) \qquad (\theta_1, \theta_2 = \text{const}) \tag{1.1}$$

resulting from the action of vertical and horizontal unit forces, respectively, applied at the origin ($x = 0$, $y = 0$). If the foundation is elastic, in virtue of the reciprocal theorem, the displacement $v_1^*(x)$ can be determined as horizontal displacements due to a unit vertical force.

Once the function v_0^* and v_1^*, are known, the problem can be formulated in terms of an integral Eq.

$$\int_0^l [\theta_1 v_0(x-s) + k\theta_2 v_1(x-s)]\, p(s)\, ds = f(x) \qquad (0 \leqslant x \leqslant l,\ f(x) = \delta + \theta x + g(x)) \tag{1.2}$$

Here δ and θ are the settlement and the angle of rotation of the punch, respectively.

As in previous work [1 and 2], we will assume that the influence functions may be represented by Fourier integrals, i. e.

$$v_0(x) = \frac{1}{\pi}\int_0^\infty \frac{\varphi_0(t)}{t}\cos tx\,dt, \qquad v_1(x) = \frac{1}{\pi}\int_0^\infty \frac{\varphi_1(t)}{t}\sin tx\,dt, \qquad \varphi_0(0)=0 \quad (1.3)$$

However, unlike [1 and 2], we will assume a more general asymptotic representation

$$\varphi_m(t) = t^\nu[1+O(t^{-\varepsilon})] \qquad (0\leqslant\nu<1,\ \varepsilon>0,\ m=0,1) \qquad (1.4)$$

The reason for the generalization is the desire to include in the general theory foundations in the form of a half-space with an elastic modulus whose variation is given by

$$E = E_\nu y^\nu \qquad (0\leqslant\nu<1) \qquad (1.5)$$

but with constant value of Poisson's ratio μ. Utilizing the results of [4] and taking into account the reciprocal theorem, we find that

$$v_0(x) = \frac{1}{\nu|x|^\nu}, \quad v_1(x) = \frac{\operatorname{sgn} x}{|x|^\nu}; \quad \theta_1 = \frac{(1-\mu^2)\gamma C}{(1+\nu)E_\nu}\sin\frac{\pi\gamma}{2},$$
$$\theta_2 = -\frac{(1-\mu^2)C}{\nu E_\nu}\cos\frac{\pi\gamma}{2} \qquad (1.6)$$
$$\left(C = \frac{\Gamma[1+{}^1/_2(1+\nu+\gamma)]\,\Gamma[1+{}^1/_2(1+\nu-\gamma)]}{2^{-1-\nu}\pi\Gamma(2+\nu)}, \quad \gamma = \left[(1+\nu)\left(1-\frac{\nu\mu}{1-\mu}\right)\right]^{1/2}\right)$$

In this case, the integral Eq. (1.2) takes the form

$$\int_0^l \left[\frac{\theta_1}{\nu} + \theta_2 k\operatorname{sgn}(x-s)\right]\frac{p(s)\,ds}{|x-s|^\nu} = f(x) \qquad (1.7)$$

Letting $\nu\to 0$ and noting (1.6), we obtain the integral equation for the plane contact problem including Coulomb friction forces for the usual half-space [1 and 2]

$$\int_0^l \left[\frac{2(1-\mu^2)}{\pi E}\ln\frac{1}{|x-s|} + \frac{(1+\mu)(1-2\mu)}{2E}\operatorname{sgn}(x-s)\right]p(s)\,ds = f(x) + \text{const} \quad (1.8)$$

Taking note of (1.6) and of Formulas 3.761 in [5], we see that the elastic half-space whose modulus is of the form (1.5) is a particular case

$$\varphi_0(t) = \pi\Gamma^{-1}(1+\nu)\sec{}^1/_2\,\nu\pi t^\nu, \qquad \varphi_1(t) = \pi\Gamma^{-1}(\nu)\operatorname{cosec}{}^1/_2\,\nu\pi t^\nu$$

of the previously introduced linearly deforming foundation characterized by Formulas (1.1), (1.3) and (1.4).

Utilizing the representation (1.3) and Formulas 3.761 in [5], we can write

$$\pi v_0(x) = \Gamma(\nu)\cos{}^1/_2\,\nu\pi\,|x|^{-\nu} - \pi l_0(x), \qquad \pi v_1(x) = \Gamma(\nu)\sin{}^1/_2\,\nu\pi\,|x|^{-\nu}\operatorname{sgn} x - \pi l_1(x) \ (1.9)$$

Moreover, the functions $l_1(x)$ which are defined by the integrals

$$\begin{matrix} l_0(x) \\ l_1(x)\end{matrix} = \frac{1}{\pi}\int_0^\infty \left[t^\nu - \begin{matrix}\varphi_0(t) \\ \varphi_1(t)\end{matrix}\right]\begin{matrix}\cos tx \\ \sin tx\end{matrix}\ \frac{dt}{t} \qquad (1.10)$$

will be, in view of the asymptotic representation in (1.4), at least continuous functions. Taking into account (1.9), the integral Eq. (1.2) may be written in the form

$$\frac{\Gamma(\nu)}{\pi}\int_0^l \left[\theta_1 \cos\frac{\nu\pi}{2} + k\theta_2 \sin\frac{\nu\pi}{2}\operatorname{sgn}(x-s)\right] p(s)\frac{ds}{|x-s|^\nu} =$$
$$= f(x) + \int_0^l [\theta_1 l_0(x-s) + k\theta_2 l_1(x-s)]\, p(s)\, ds \qquad (0 \leqslant x \leqslant l) \tag{1.11}$$

From the above it follows that if the inversion formula for integral equations of the type (1.7) were known then the integral Eq. of the first kind (1.11), which is here under consideration, could be reduced to an equation of the second kind with a continuous kernel.

Here it is helpful to note that Arutiunian and Manukian [3], in solving the plane contact problem while taking into account friction for a nonlinearly deforming foundation (taking into account creep), obtained the integral Eq.

$$\int_0^l \frac{[a_1 \operatorname{sgn}(x-s) + a_2]^{1-\nu}}{|x-s|^\nu} p(s)\, ds = f(x) \tag{1.12}$$

which is similar to the one obtained here (for an inhomogeneous elastic half-space). Eq. (1.12) was solved by the authors by a combination of particular procedures, obtaining the solution for a special right-hand side (equal to unity) and utilizing the general formulas of Krein [6].

Galin [7] solved integral Eq. (1.7) by reducing it to a boundary value problem in analytic functions, and obtained the solution in terms of the principal value of certain integrals. A solution of the same form and utilizing the same method but for a clever generalization of the equation was obtained by K. D. Sakaliuk [8].

Below, an entirely different method from those of the above mentioned authors is utilized, and two procedures are given for solving an integral equation of a somewhat more general form than (1.7) and (1.12).

2. Consider the integral Eq.

$$\int_0^l \frac{[a \operatorname{sgn}(x-y) + b]^\sigma}{|x-y|^\nu} p(y)\, dy = f(x) \qquad (0 \leqslant x \leqslant l,\ 0 \leqslant \nu < 1) \tag{2.1}$$

By a change of variables

$$x = l\xi, \qquad y = l\eta, \qquad l^{1-\nu} p(l\xi) = \varphi(\xi) \tag{2.2}$$

the above is reduced to the form

$$\int_0^1 \frac{[a \operatorname{sgn}(\xi-\eta) + b]^\sigma}{|\xi-\eta|^\nu} \varphi(\eta)\, d\eta = f(l\xi) \qquad (0 \leqslant \xi \leqslant 1) \tag{2.3}$$

It turns out that for this integral Eq. there are (with proper weight functions) a pair of orthonormal Schmidt systems [9] which are very simply related to the Jacobi polynomials $P_m^{\alpha,\beta}(x)$. This mathematical fact is expressed by the relation

$$\int_0^1 \frac{[a \operatorname{sgn}(\xi-\eta) + b]^\sigma P_m^{\alpha-1,\, \nu-\alpha}(1-2\eta)}{|\xi-\eta|^\nu \eta^{1-\alpha}(1-\eta)^{\alpha-\nu}} d\eta = \frac{A\pi(\nu)_m P_m^{\nu-\alpha,\, \alpha-1}(1-2\xi)}{m! \sin\pi\nu\, (a+b)^{-\sigma}} \tag{2.4}$$

$$\left(A = \sqrt{1 - 2\cos\pi\nu a_*^{\ \sigma} + a_*^{\ 2\sigma}}, \quad a_* = (b-a)(a+b)^{-1}, \ \ \operatorname{Re} A > 0,\ 0 < \operatorname{Re}\nu < 1\right)$$

The remaining parameters are related by

$$\frac{\sin\pi(\alpha-\nu)}{\sin\pi\alpha}=a_*^{\sigma}, \qquad \text{or} \qquad \alpha=\frac{1}{\pi}\sin^{-1}\frac{\sin\pi\nu}{A} \tag{2.5}$$

Here, that single-valued branch of the arcsine is chosen for which $0<\operatorname{Re}\alpha<1$. Proof of (2.4) is based on Formula

$$\frac{[a\operatorname{sgn}(\xi-\eta)+b]^{\sigma}}{(a+b)^{\sigma}|\xi-\eta|^{\nu}}=\frac{\xi^{\alpha-\nu}\eta^{1-\alpha}\Gamma(\alpha)}{\Gamma(\nu)\Gamma(\alpha-\nu)}I(\xi,\eta) \tag{2.6}$$

$$I(\xi,\eta)=\int_0^{\min(\xi,\eta)}\frac{s^{\nu-1}\,ds}{(\xi-s)^{\alpha}(\eta-s)^{1+\nu-\alpha}} \qquad (\operatorname{Re}(1+\nu-\alpha)<1)$$

Let us prove this formula. For this purpose, we evaluate the integral contained therein, taking separately the case $\xi<\eta$ and the case $\eta<\xi$. Obvious changes of variables and utilization of the known [5] integral representation of Gauss' hypergeometric function lead to Formula

$$I(\xi,\eta)=\begin{cases}\dfrac{\Gamma(\nu)\Gamma(1-\alpha)}{\Gamma(\nu-\alpha+1)}\,\dfrac{F(1-\alpha+\nu,\ \nu;\ 1-\alpha+\nu;\ \xi/\eta)}{\xi^{\alpha-\nu}\eta^{1-\alpha+\nu}} & (\xi<\eta)\\[2ex] \dfrac{\Gamma(\nu)\Gamma(\alpha-\nu)}{\Gamma(\alpha)}\,\dfrac{F(\alpha,\ \nu;\ \alpha;\ \eta/\xi)}{\eta^{1-\alpha}\xi^{\alpha}} & (\eta<\xi)\end{cases} \tag{2.7}$$

Upon expressing Gauss' functions in the above in terms of elementary functions ([5], p. 1054) (and upon straightforward transformation into gamma functions) we obtain

$$I(\xi,\eta)=\frac{\Gamma(\nu)\Gamma(\alpha-\nu)}{\Gamma(\alpha)}\frac{\xi^{\nu-\alpha}\eta^{\alpha-1}}{|\xi-\eta|^{\nu}}\begin{cases}1, & \eta<\xi\\ \sin\pi(\alpha-\nu)\operatorname{cosec}\pi\alpha, & \xi<\eta\end{cases} \tag{2.8}$$

On the other hand, the following Eq. clearly holds

$$\frac{[a\operatorname{sgn}(\xi-\eta)+b]^{\sigma}}{(a+b)^{\sigma}|\xi-\eta|^{\nu}}=\frac{1}{|\xi-\eta|^{\nu}}\begin{cases}1, & \eta<\xi\\ a_*^{\sigma}, & \xi<\eta\end{cases}$$

Taking into account the above together with (2.7) and (2.4), we arrive at the proof of (2.6).

Let the left-hand side of (2.4) be denoted by $J(x)$. Then, from (2.6), we can write

$$J(\xi)=\frac{\Gamma(\alpha)(a+b)^{\sigma}\xi^{\alpha-\nu}}{\Gamma(\alpha)\Gamma(\alpha-\nu)}\int_0^1\frac{P_m^{\alpha-1,\ \nu-\alpha}(1-2\eta)}{(1-\eta)^{\alpha-\nu}}\,d\eta\int_0^{\min(\xi,\eta)}\frac{s^{\nu-1}\,ds}{(\xi-s)^{\alpha}(\eta-s)^{1-\alpha+\nu}}$$

Now, inverting the order of integration (the legitimacy of this step follows from the bounds on the parameter) in a similar manner to Copson's method, we obtain

$$J(\xi)=\frac{\Gamma(\alpha)(a+b)^{\sigma}\xi^{\alpha-\nu}}{\Gamma(\nu)\Gamma(\alpha-\nu)}\int_0^{\xi}\frac{s^{\nu-1}}{(\xi-s)^{\alpha}}I(s)\,ds \tag{2.9}$$

Here

$$I(s)=\int_s^1\frac{P_m^{\alpha-1,\ \nu-\alpha}(1-2\eta)\,d\eta}{(1-\eta)^{\alpha-\nu}(\eta-s)^{1-\alpha+\nu}} \tag{2.10}$$

In evaluating the last integral, the following relations are used:

$$m!\,P_m^{\alpha,\beta}(1-2x) = (1+\alpha)_m F(\alpha+\beta+m+1,\ -m;\ 1+\alpha;\ x)$$

$$\int_0^1 \frac{t^{\gamma-1}}{(1-t)^{1-\varepsilon}} F(\alpha,\beta;\gamma;tz)\,dt = \frac{\Gamma(\gamma)\Gamma(\varepsilon)}{\Gamma(\gamma+\varepsilon)} F(\alpha,\beta;\gamma+\varepsilon;z) \tag{2.11}$$

$$\Gamma(1-\alpha)F(\nu+m,\ -m;\ \alpha;\ z) =$$
$$= \Gamma(1-\alpha-m)(1-\alpha+\nu)_m F(\nu+m,\ -m;\ 1-\alpha+\nu;\ 1-z)$$

These relations are, respectively, corollaries of formulas 8.962, 7.512(8) and 9.131) in [5].

Using the first formula in (2.11) and making the substitution $\eta = s + t(l-s)$ in (2.10), we obtain

$$I(s) = \frac{(\alpha)_m}{m!}\int_0^1 \frac{F(\nu+m,\ -m;\ \alpha;\ 1+t(1-s))}{t^{1-\alpha+\nu}(1-t)^{\alpha-\nu}}\,dt$$

Utilization of the third formula in (2.11) and the substitution $1-t=\tau$ lead to

$$I(s) = \frac{(-1)^m}{m!}(1-\alpha+\nu)_m \int_0^1 \frac{\tau^{\nu-\alpha}}{(1-\tau)^{1-\alpha+\nu}} F(\nu+m,\ -m;\ 1-\alpha+\nu;\ (1-s)\tau)\,d\tau$$

Whence it follows from application of the second and third formulas in (2.11) that

$$I(s) = \frac{\pi(1-\alpha+\nu)_m(\nu)_m}{m!^2\sin\pi(\alpha-\nu)} F(\nu+m,\ -m;\ \nu;\ s)$$

Substituting the obtained result into (2.9) and letting $s=\xi t$, we obtain

$$J(\xi) = \frac{\pi\Gamma(\alpha)(\nu)_m(1-\alpha+\nu)_m(a+b)^\sigma}{m!^2\Gamma(\nu)\Gamma(\alpha-\nu)\sin\pi(\alpha-\nu)}\int_0^1 \frac{F(m+\nu,\ -m;\ \nu;\ t\xi)}{(1-t)^\alpha}\,dt$$

Finally, use of the second and first formulas in (2.11) yields

$$J(\xi) = \frac{\pi(a+b)^\sigma(\nu)_m}{m!\sin\pi\alpha} P_m^{\nu-\alpha,\ \alpha-1}(1-2\xi)$$

By means of analytic continuation, the requirement that Re $(1+\nu-\alpha)<1$, which was made in (2.6) may now be eliminated.

Taking note of (2.5), we conclude that the right-hand side of the last relation coincides with the right-hand side of (2.4), completing its proof.

The proven relation permits the use of the following procedure for solving integral Eq. (2.3). Using the orthogonality property of Jacobi polynomials, the right-hand side of (2.3) is expanded into the series

$$f(l\xi) = \sum_{n=0}^{\infty} c_n P_n^{\nu-\alpha,\ \alpha-1}(1-2\xi)$$

Then the solution of integral Eq. (2.3), in view of (2.4) will have the form

$$\varphi(\xi) = \sum_{n=0}^{\infty} \frac{c_n \sin\pi\nu\, n!}{A\pi(a+b)^\sigma(\nu)_n}\ \frac{P_n^{\alpha-1,\ \nu-\alpha}(1-2\xi)}{\xi^{1-\alpha}(1-\xi)^{\alpha-\nu}}$$

Such a form of the solution might be more convenient than a reduction to quadrature.

3. Formula (2.5), obtained in the preceding section also permits the reduction to quadrature of integral Eq. (2.2). The previously mentioned Copson's method [10] quickly leads to the desired result. Indeed, from (2.6), the left-hand side of (2.2) may be rewritten as

$$\frac{\Gamma(\alpha)(a+b)^{\sigma}}{\Gamma(\nu)\Gamma(\alpha-\nu)}\xi^{\alpha-\nu}\int_0^{\xi}\frac{s^{\nu-1}}{(\xi-s)^{\alpha}}ds\int_s^1\frac{\eta^{1-\alpha}\varphi(\eta)}{(\eta-s)^{1-\alpha+\nu}}d\eta=f(l\xi) \tag{3.1}$$

Thus, the integral Eq. under consideration is reduced to two Abel type iterated integrals. Employing the known (see, for example, [8]) inversion formulas

$$\int_0^x\frac{\chi(t)\,dt}{(x-t)^{\mu}}=g(x),\qquad \chi(x)=\frac{\sin\mu\pi}{\pi}\frac{d}{dx}\int_0^x\frac{g(t)\,dt}{(x-t)^{1-\mu}}$$

$$\int_x^1\frac{\psi(t)\,dt}{(t-x)^{\mu}}=q(x),\qquad \psi(x)=-\frac{\sin\mu\pi}{\pi}\frac{d}{dx}\int_x^1\frac{q(t)\,dt}{(t-x)^{1-\mu}}$$

we obtain from (3.1)

$$\varphi(\xi)=-\frac{B}{\xi^{1-\alpha}}\frac{d}{d\xi}\int_{\xi}^1\frac{s^{1-\nu}\,ds}{(\xi-s)^{d-\nu}}\frac{d}{ds}\int_0^s\frac{f(tl)\,t^{\nu-\alpha}}{(s-t)^{1-\alpha}}dt\qquad\left(B=\frac{\sin\pi\alpha\Gamma(\nu)(a+b)^{-\sigma}}{\pi\Gamma(\alpha)\Gamma(1-\alpha+\nu)}\right)$$

Hence, taking into account (2.2), the solution of integral Eq. (2.1) takes the form

$$p(x)=-\frac{B}{x^{1-\alpha}}\frac{d}{dx}\int_x^l\frac{s^{1-\nu}\,ds}{(s-x)^{\alpha-\nu}}\frac{d}{ds}\int_0^s\frac{t^{\nu-\alpha}f(t)\,dt}{(s-t)^{1-\alpha}}$$

or, upon integrating by parts

$$p(x)=\frac{B}{x^{1-\alpha}}\left(\frac{\Phi(l)}{(l-x)^{\alpha-\nu}}-\int_x^l\frac{\Phi'(s)\,ds}{(s-x)^{\alpha-\nu}}\right) \tag{3.2}$$

$$\left(\Phi(x)=x^{1-\nu}\frac{d}{dx}\int_0^x\frac{t^{\nu-\alpha}f(t)\,dt}{(x-t)^{1-\alpha}},\qquad B=\frac{\sin\pi\alpha\Gamma(\nu)(a+b)^{-\sigma}}{\pi\Gamma(1-\alpha+\nu)\Gamma(\alpha)}\right)$$

The parameter α is as defined in (2.5).

This result may also be obtained in a different way. In this method, (2.4) is employed in order to obtain a solution of integral Eq. (2.1) with the right-hand side equal to unity, and then we apply Krein's formulas [6]. Such a procedure is used in [3] in order to solve (1.12), but there, a different procedure is used to obtain the solution to the problem with the right-hand side equal to unity.

Clearly, in order to solve integral Eq. (1.7) of the plane contact problem in which friction forces are taken into account and the half-space has an elastic modulus given by (1.5), we have to set $\sigma=1$, $a=k\theta_2$ and $b=\theta_1\nu^{-1}$, in Eqs. (3.2) and (2.5), with θ_j as defined by the Formulas in (1.6). In order to obtain the solution to integral Eq. (1.12), the problem of Arutiunian and Manukian, we let $\sigma=1-\nu$, $a=a_1$ and $b=a_2$.

4. The relation (2.4) obtained in Section 2, which may also be written in the form

$$\int_0^l \frac{[a\,\mathrm{sgn}\,(x-y)+b]^\sigma P_m^+(y)}{|x-y|^\nu y^{1-\alpha}(l-y)^{\alpha-\nu}}\,dy = \mu_m P_m^-(x) \tag{4.1}$$

$$\left(\sigma=1,\ \mu_m=\frac{A\pi(\nu)_m}{m!\sin\pi\nu},\quad P_m^+(x)=P_m^{\alpha-1,\,\nu-\alpha}(1-2x/l),\ P_m^-(x)=P_m^{\nu-\alpha,\,\alpha-1}(1-2x/l)\right)$$

permits the use of an effective approximate method in solving integral Eq. (1.11), at the same time solving the contact problem with friction for a linearly deforming foundation. For this purpose, the functions in (1.10) are approximated by polynomials

$$l_0(x)\approx\sum_{j=0}^n a_j^0x^{2j},\qquad l_1(x)\approx\sum_{j=0}^n a_j^1x^{2j+1} \tag{4.2}$$

If the functions in (1.10) are analytic, then Expressions in (4.2) can be truncated Maclaurin series.

The function $f(x)$ in (1.11) is expanded in terms of Jacobi polynomials, and a solution is sought in the form of a similar series, i.e.

$$f(x)=\sum_{m=0}^\infty A_mP_m^-(x),\qquad p(y)=\sum_{m=0}^\infty\frac{P_m^+(y)X_m}{y^{1-\alpha}(l-y)^{\alpha-\nu}} \tag{4.3}$$

Substitute (4.2) and (4.3) into (1.11); multiply the resultant equation by

$$P_k^-(x)\,x^{\nu-\alpha}(l-x)^{\alpha-1}\ (k=0,1,2\ldots)$$

and integrate over the interval $(0, l)$. As a result of orthogonality of the Jacobi polynomials, we obtain

$$\lambda_k\mu_kX_k=\sum_{m=0}^{N-k}X_m\sum_{j=k+m}^N C_jB_{mj}^k+\lambda_kA_k\left(k\leqslant N=2n+1,\ X_k=\frac{A_k}{\mu_k},\quad k>N\right) \tag{4.4}$$

Here

$$C_{2j}=\theta_1a_j^0,\qquad C_{2j+1}=k\theta_2a_j^1\qquad(j=0,1,2,\ldots n),$$

$$\lambda_m=\int_0^l\frac{[P_m^-(x)]^2\,dx}{x^{\alpha-\nu}(l-x)^{1-\alpha}}=\frac{l^\nu}{2}\,\frac{\Gamma(1-\alpha+\nu+m)\,\Gamma(\alpha+m)}{m!\,(\nu+2m)\,\Gamma(\nu+m)}$$

$$B_{mj}^k=0\qquad(j-k<m)$$

$$B_{mj}^k=\frac{(-1)^{m+k}}{l^{2-j}}\sum_{r=m}^{j-k}\frac{(-1)^r j!\,\Gamma(1-\alpha+\nu+j-r)\,\Gamma(\alpha+k)\,\Gamma(1-\alpha+\nu+m)}{k!\,m!\,(j-k-r)!\,(r-m)!\,\Gamma(1+\nu+j+k-r)\,\Gamma(1+\nu+m+r)}$$

$$(j-k>m)$$

The last formula is easily obtained from

$$B_{mj}^k=\int_0^l\int_0^l\frac{(x-y)^jx^{\nu-\alpha}y^{\alpha-1}}{(l-x)^{1-\alpha}(l-y)^{\alpha-\nu}}\,P_k^-(x)\,P_m^+(y)\,dx\,dy\,.$$

if we take into account Formula 7.391(4) in [5]. Thus, the problem is reduced to the solution of a system of algebraic Eqs. (4.4) with a triangulated coefficient matrix.

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Translated by H. H.